This is a closed-book, closed-notes exam. The only items you are allowed to use are writing implements. Mark each sheet of paper you use with your name and clearly indicate the problem number.

The max number of points per question is indicated in square brackets after each question. The sum of the max points for all the questions is 53, but note that the max exam score will be capped at 50 (i.e., there are 3 bonus points, but you cant score more than 100%). Partial credit will be awarded, so show your work!

You have exactly 60 minutes to complete this exam. Keep your answers clear and concise while complete. Good luck!

1. (4 points) Is "This circle is both red and blue" a statement? Explain why or why not.

Solution: Yes: it is a sentence that is either true or false but not both. In this case it is probably read as false, unless we're thinking of a striped circle!

2. (8 points) Compute $GCD(420, 72)$ using either unique prime factorization or Euclid's algorithm.

Solution: Using prime factorization:

$$\begin{aligned} 420 &= 2^2 * 3 * 5 * 7 \\ 72 &= 2^3 * 3^2 \end{aligned}$$

420 and 72 share the prime factors 2, 2, and 3, so the greatest common divisor is $2^2 * 3 = 12$.

Using Euclid's algorithm:

$$\begin{aligned} GCD(420, 72) &= GCD(72, 420 \bmod 72) = GCD(72, 60) \\ &= GCD(60, 72 \bmod 60) = GCD(60, 12) \\ &= GCD(12, 60 \bmod 12) = GCD(12, 0) \\ &= 12 \end{aligned}$$

3. (8 points) Given $D = \{2, 3, 4\}$,

   (a) (4 points) List the elements of $D \times D$.

   (b) (4 points) List the elements of $\{(m, n) \in D \times D \mid m \bmod n = 1\}$.

Solution: Part (a):

$$\begin{aligned} D \times D = \{&(2,2), (2,3), (2,4), \\ &(3,2), (3,3), (3,4), \\ &(4,2), (4,3), (4,4)\} \end{aligned}$$

Part (b):

$$
\begin{aligned}
2 \bmod 2 &= 0\\
2 \bmod 3 &= 2\\
2 \bmod 4 &= 2\\
3 \bmod 2 &= 1\\
3 \bmod 3 &= 0\\
3 \bmod 4 &= 3\\
4 \bmod 2 &= 0\\
4 \bmod 3 &= 1\\
4 \bmod 4 &= 0
\end{aligned}
$$

So $\{(m, n) \in D \times D \mid m \bmod n = 1\} = \{(3, 2), (4, 3)\}$

4. (10 points) Given the following premises, determine which, if any, of the conclusions listed make the argument valid. Explain your reasoning for your answer.

   Premises:

$$
\begin{aligned}
&p \rightarrow q \wedge \sim r\\
&q \rightarrow p \vee \sim r\\
&q \vee \sim r
\end{aligned}
$$

   Possible Conclusions: p $\quad q \rightarrow p$ $\quad \sim r$

Solution: For an argument to be valid, its conclusion must follow from its premises. That is, whenever the premises are true, the conclusion must also be true.

Constructing a truth table for the given premises shows that $\sim r$ is the only conclusion that is true in every case that the premises are true.

5. (8 points) Given the statement "If I drink enough coffee, I will never need to sleep again",
   (a) (4 points) Write the negation of the statement.
   (b) (4 points) Write the contrapositive of the statement.

Solution: The negation of $p \rightarrow q$ is $p \wedge \sim q$, so the negation of this statement is "I drink enough coffee and I still need to sleep again".

The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$, so the contrapositive of this statement is "If I need to sleep again, then I did not drink enough coffee."

(Note: Please do not take anything written on a discrete math exam as health advice.)

6. (15 points) Prove that for every integer n, n^2 can be written as $3k$ or $3k + 1$.

   Hint: divide n by 3 using the Quotient-Remainder Theorem.

Solution: By the Q-R theorem, let q and r be integers where $n = 3q + r$ and $0 \leq r < 3$. Since there are only three possible values for r, we have three cases:

Case 1: $r = 0$, so $n = 3q$.
Then $n^2 = (3q)^2 = 9q^2 = 3(3q^2)$.
Let $k = 3q^2$. k is an integer because it is the product of integers.
Thus $n^2 = 3k$.

Case 2: $r = 1$, so $n = 3q + 1$.
Then $n^2 = (3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$.
Let $k = 3q^2 + 2q$. Again, k is an integer.
Thus $n^2 = 3k + 1$.

Case 3: $r = 2$, so $n = 3q + 2$.
Then $n^2 = (3q + 2)^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1 = 3(3q^2 + 4q + 1) + 1$.
Let $k = 3q^2 + 4q + 1$. Wow, k is also an integer here!
Thus $n^2 = 3k + 1$.
Therefore, for all integers n, there exists an integer k where $n^2 = 3k$ or $n^2 = 3k + 1$.